Exhaustion of Ductility and Ultimate Fracture

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Introduction

The fracture or separation at the Ultimate Load plays an important part in engineering design and analysis. It is the ultimate point at which we will have irreversible damage to the structure in question and from which collateral damage may ensue. Invariably our designs carry an implicit safety factor w.r.t. this point either by means of the analytical tools we use or by the experimental facts we elicit by examining failures.

The application of linear/ quasi Linear Elastic Fracture Mechanics (LEFM) is well established, Griffith[1], Rice[2]. The LEFM has been modified to account for limited plastic behavior by Irwin [3]. When plastic behavior is a significant factor, the behavior is referred to as ductile Fracture Mechanics [4]. The mechanics of fracture has also been applied to the tearing of Vulcanized rubber by Rivlin and Thomas [5]. This study concerned large nonlinear incompressible strains that were modeled by Mooney Rivlin Materials, Mooney [6].

In the domain of structural mechanics, not every metal structure is composed of a specific material and/or subjected to environmental conditions as to result in LEFM. In contrast, when a structure is loaded to its ultimate load, it will fail by what we shall call Ultimate Fracture Mechanics (UFM). Before we arrive at this point, we will have passed through the limit load and at that point, the assumption of purely plastic behavior and small strain is justified by both theory and experiment [7,8]. We shall assume that at Ultimate Fracture, we will only have plastic deformation. Marcal [9] adapted the Mooney Rivlin material for optimal plastic design by using the appropriate values for metal in a Finite Element Analysis (FEA). Here we use the same FEA [10] and assume a plastic work hardening material with an initial yield of 10,000 psi. We can then obtain estimates of the plastic J-integral by considering a balance of the surface work due to fracture and the difference in potential energies before and after fracture. In this case the behavior is assumed to be purely plastic as defined by the Mooney Rivlin Material for our assumed metal.

McClintock [11] has suggested a failure or damage mechanism where holes are formed in the plastic material. This damage criteria has been adopted by the studies in ductile Fracture to better define the plastic zone. In our formulation, we need not specify the exact mode of the failure mechanism, we only need to balance the work of fracture to the difference of potential energy before and after fracture has occurred.

Theoretical Considerations

We now review the equations for Fracture that we can use for the plastic fracture. First we define an associated nonlinear elastic structure defined by the Mooney Rivlin metal material. This Mooney Rivlin material is incompressible and is a function of the equivalent stress, sig_equiv. It is thus defined by the normal flow equations of plasticity.
Let J be the surface constant of Fracture, 
A be the surface area of the crack,
And W(s), W(e) be the starting and ending potential energy respectively.

Then for a small incremental crack growth dA, we equate work at the crack surface to the change in internal energy of the associated structure,

\[ J \cdot dA = d(W(s) - W(e)) \] (1a)

\[ J = \frac{d(W(s) - W(e))}{dA} \] (1b)

As usual J is defined numerically with the aid of experiment to determine the ultimate load point.

When a material is stretched, its cross-sectional area contracts. This requires that its stiffness modulus increases just to maintain equilibrium. At some point the increase in material stiffness cannot compensate for the loss in cross-sectional area and we have geometric instability. It is this instability that will prevent us from reaching the full potential in our large displacement analysis with rubber-like materials. These materials are usually modeled with the Mooney Rivlin Materials. A special form of hyperelastic materials where the change in volume is set to zero [6].

In MPACT, the Strain Energy is written as

\[ W = K_0 (B(1) - 3) + K_1 (B(2) + 3) \] (2)

Where \( K_0 \) and \( K_1 \) are constants and
\( B(1), B(2) \) are the first and second invariant of Finger’s strain tensor.

We will first consider converting material data from a tensile test to the constants \( K_0, K_1 \). In a tension test, we consider the principal stress \( T \) and the corresponding principal length ratio \( \lambda \). These are related to the material constants by

\[ T = 2\lambda(\lambda - 1/(\lambda^2))(K_0 - K_1/\lambda) \] (3a)

Moving terms to the right

\[ T/2\lambda(\lambda - 1/(\lambda^2)) = K_0 - 1/\lambda \cdot K_1 \] (3b)

We note that if we plot the term on the right against \( 1/\lambda \),

The intercept with the axis is \( K_0 \) and the slope of the curve is \(-K_1\).

Fig. 1 shows the plot of stress vs inverse of strain used to obtain the Mooney Rivlin constants \( K_0, K_1 \) From which we obtained the constants,

\( K_0 = 250 \) kips and \( K_1 = 5.56 \) kips
Fig. 2 shows the more traditional plot of stress vs strain. Which may be interpreted as the specific potential energy while noting that it is concave and so is suitable for a nonlinear elastic analysis of the associated structure.

![Mooney Rivlin Constants](image)

Fig. 1 Stress vs inverse strain eq. (3b)

![Stress-Strain](image)

Fig. 2 Plastic Stress Strain
Note on the consequences of assuming that the specific energy is restricted to a function of the equivalent stress

We will examine the nature of the Mooney Rivlin material, expressed below

\[ S_{ij} = \frac{dW}{dU_{ij}} - pB_{ij} \quad (4a) \]

Where \( S_{ij} \) is the Kirchhoff’s stress tensor

\( U_{ij} \) is Almansi’s strain tensor

\( p \) is an undetermined factor

\( B_{ij} \) is the Kronecker delta

Because \( W \) is a function of \( U_{ij} \), we can write

\[ \{S\} = [D(U_{ij})][U] \quad (4b) \]

Where \([D]\) is the stress strain matrix.

This tells us that the solution is essentially a total solution. However, we have imposed a constraint that the energy \( W \) is a function of the equivalent stress and the condition of incompressibility also implies the satisfaction of the flow rule. The best that we can do to satisfy the incremental plastic flow rule is to carry out the solution incrementally and to let the implied constraints satisfy themselves as best they can. Also to repeat the solution at each increment to allow Residual corrections.

Test Case, Ultimate Fracture of a built-in cracked beam

We study the case of a built in beam with dimensions 40 X 10 X10 ins.

The beam is shown in Fig. 3.
A series of analysis is made with the built in crack starting from the top to y=8 ins.
The crack is then allowed to extend downwards 1 in. at a time.
Fig. 4 gives the result of the surface constant J calculated for the crack positions.
To round-off the presentation, we include contours of the equivalent stress and the equivalent plastic strain component $\varepsilon_{\text{xx}}$.

Fig. 5 shows one of the weaknesses of using the Mooney Rivlin material for cracks, that is, the analysis cannot keep the stress within the plastic limit. Fortunately, the crack line is thin and the stress excursion will be limited in terms of the overall structure.
Because of the normal flow rule, we can interpret the $\varepsilon_{xx}$ component as also representing the equivalent strain. Fig. 6 shows that there is plastic deformation throughout the structure at Ultimate Fracture. That is also why we had to assume a stress strain relation where a plastic strain would exist at a small stress. We observe that a high region of deformation is generated around the crack tip. However, there is little we can do by examining this zone closely. The prediction of fracture comes out of a systemic view by balancing total work and not by concentration on mathematical singularities.

**Conclusions and Suggestions for Future Work**

A method has been developed for fracture at the ultimate load by assuming only plastic deformation at that point. The method makes use of an associated nonlinear elastic structure that is incompressible and based on the equivalent stress. By its assumptions, the associated structure is represented by the stress plastic strain curve of a Mooney Rivlin material.

The fracture surface constant $J$ is calculated numerically by estimating the change in potential energy w.r.t. crack length with the aid of a FEM code. This fracture surface constant $J$ is not fully defined until it is correlated with the actual conditions of Ultimate Fracture of a test specimen.

The method has been demonstrated for a built-in beam subjected to a surface vertical pressure.

The Ultimate Fracture is an important event in the behavior of a structure. Much more work is required to fully substantiate the method proposed and demonstrated here. It is interesting to speculate that because it is such a common event in any test, experimental results are already available to help us in this further study.

**References.**


[9] Marcal, PV, ‘The finite element method applied to optimal plastic design’, to be published, 2018

See,
